Indian Statistical Institute, Bangalore

B. Math. Second Year

Second Semester - Computer Science II

Mid-Semester Exam | Duration: 3 hours | Date : Feb. 20, 2017

Answer all the questions. Max Marks: 30

1. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. For $x \in (a, b)$ and $h > 0$ define the difference quotient
$$D_h f(x) = \frac{f(x+h) - f(x)}{h}, \tag{DQ}$$
show that there exists a constant $C > 0$ such that
$$|D_h f(x) - f'(x)| \leq Ch$$
where $'$ denotes differentiation. Assuming f to be complex, show that
$$C_h f(x) = \frac{Im\, f(x+ih)}{h} \tag{CQ}$$
is another approximation to $f'(x)$. Let $x_0 \in (a, b)$ and $f(x) = 1/x$ write down the analytical expressions for $D_h f(x_0)$ and $C_h f(x_0)$. Which of the formulas (DQ or CQ) would be your choice for computing the derivative of a function numerically on the computer? [15]

2. Assume the theorem: Let $\Omega \subset \mathbb{R}$ be closed and $f : \Omega \to \mathbb{R}$ be a continuously differentiable function satisfying $f(\Omega) \subset \Omega$ and for $x, y \in \Omega$
$$|f(x) - f(y)| \leq L\, |x - y| \;\text{ with }\; 0 < L < 1.$$
Then, there exists a unique $z \in \Omega$ satisfying
$$f(z) = z \tag{FP}$$
which is called the fixed point of f. The iteration ($n \geq 1$)
$$x_n = f(x_{n-1}) \tag{FPI}$$
for any given $x_0 \in \Omega$ converges to the z satisfying (FP). Now answer the following.

   (a) Let $\Omega = [0, 1]$ and $f(x) = ax + b$ in (FPI). Given a $x_0 \in \Omega$ write down the formula for x_n and discuss the convergence as $n \to \infty$ for
$$|a| < 1,\; b \in \mathbb{R},$$
$$|a| = 1,\; b \in \mathbb{R},$$
$$|a| > 1,\; b \in \mathbb{R}.$$

(b) How can one determine L for an f that is continuously differentiable. Given a $x_0 \in \Omega$ consider (FPI) for $j \geq 1$ and set $e_j = x_j - x_{j-1}$. Show that

$$|e_n| \leq L^{n-1}\, e_1.$$

(c) Consider the problem of finding an $x \in [0,1]$ satisfying

$$ax^3 = (1+x)^2$$

for $a > 0$ using (FPI). We have two choices for (FP)

$$f(x) = \sqrt{a}\, x^{\frac{3}{2}} - 1$$

or

$$f(x) = a^{-\frac{1}{3}}(1+x)^{\frac{2}{3}}.$$

For $a = 8$ which one would be your first choice and why? [15]